Secondary 3 Honors
Notes 12-6: DeMoivre's Theorem

Name: ______________________

Period: ______

POWERS AND ROOTS OF COMPLEX NUMBERS

Before you can calculate the powers and roots of complex numbers, your equation needs to be in POLAR form. $r(\cos\theta + i\sin\theta)$

Discover the formula: We can use the formula for the product of complex numbers to help visualize the pattern that DeMoivre discovered. $z = r(\cos\theta + i\sin\theta)$

- First – Find z^2 by taking the product of $z \times z$

$$z \cdot z = r \cdot r\left[\cos(\theta+\theta) + i\sin(\theta+\theta)\right]$$

$$z^2 = r^2\left[\cos 2\theta + i\sin 2\theta\right]$$

- Second – Now find z^3 by taking the product of $z^2 \times z$

$$z^2 \cdot z = r^2 \cdot r\left[\cos(2\theta+\theta) + i\sin(2\theta+\theta)\right]$$

$$z^3 = r^3\left[\cos 3\theta + i\sin 3\theta\right]$$

- What would happen if we continued this pattern for powers higher than 3?

r's power = # in front of θ = n

DeMoivre's Theorem

If the polar form of a complex number is $z = r(\cos\theta + i\sin\theta)$, then for positive integers, n

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

POLAR FORM

Ex. 1: Find each power and then express the answer in rectangular form.

a. $(4+4\sqrt{3}i)^6$

$\theta = \tan^{-1}\left(\frac{\cancel{4}\sqrt{3}}{\cancel{4}}\right) = \frac{\pi}{3}$

$r = \sqrt{4^2+(4\sqrt{3})^2} = \sqrt{16+16(3)} = 8$

$\left[8\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^6$

$= 8^6\left(\cos 6\left(\frac{\pi}{3}\right) + i\sin 6\left(\frac{\pi}{3}\right)\right)$

$= 262144\,(1+0i)$

$= \boxed{262144}$

b. $(2\sqrt{3}-2i)^8$

$\theta = \tan^{-1}\left(\frac{-2}{2\sqrt{3}}\right) = \tan^{-1}\left(\frac{-\sqrt{3}}{3}\right) = \frac{5\pi}{6}$

$r = \sqrt{(2\sqrt{3})^2+(-2)^2} = \sqrt{4(3)+4} = 4$

$\left[4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\right]^8$

$= 4^8\left(\cos 8\left(\frac{5\pi}{6}\right) + i\sin 8\left(\frac{5\pi}{6}\right)\right)$

$= 65536\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= \boxed{-32768 + 56755.8i}$

$\frac{\pm 1, \pm 2, \pm 2/3}{\pm 1, \pm 3, \pm 5}$

Earlier this year, we learned about the Fundamental Theorem of Algebra to find the number of roots that a polynomial would have. We are going to combine that theorem with DeMoivre's Theorem to find the roots of complex numbers.

Distinct Roots

For a positive integer, p, the complex number $r(\cos\theta + i\sin\theta)$ has p distinct pth roots. They are found by:

$$r^{\frac{1}{p}}\left(\cos\left(\frac{\theta+2n\pi}{p}\right) + i\sin\left(\frac{\theta+2n\pi}{p}\right)\right)$$

Where $n = 0, 1, 2, \ldots, p-1$

When you solve for x in the following equations, how many solutions do you get?

$x^2 - 2 = 0$

$+2 \quad +2$

$\sqrt{x^2} = \sqrt{2}$

$x = \pm\sqrt{2}$

2 solutions

$x^4 - 256 = 0$

$+256 \quad +256$

$\sqrt[4]{x^4} = \sqrt[4]{256}$

$x = \pm\sqrt[4]{256}$

$x = \pm 4, \pm 4i$

$P=4$

a b

Ex. 2: Find all four, fourth roots of the equation: $-4-4i$

First, write the equations in polar form.

$$\theta = \tan^{-1}\left(\frac{-4}{-4}\right)+\pi = \tan^{-1}(1)+\pi = \frac{5\pi}{4} \qquad r=\sqrt{(-4)^2+(-4)^2}=\sqrt{32}$$

Then, using the equation for distinct roots, write an expression for the fourth roots. Then find all four fourth roots.

$$\sqrt{32}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$$

$P=4 \quad n=0,1,2,3$

$$\sqrt{32}^{1/4}\left[\cos\left(\frac{\frac{5\pi}{4}+2n\pi}{4}\right)+i\sin\left(\frac{\frac{5\pi}{4}+2n\pi}{4}\right)\right]$$

★ $n=0$

$$\sqrt{32}^{1/4}\left[\cos\left(\frac{\frac{5\pi}{4}+2(0)(\pi)}{4}\right)+i\sin\left(\frac{\frac{5\pi}{4}+2(0)\pi}{4}\right)\right]$$

$$\boxed{0.857+1.282i}$$

★ $n=1$

$$\sqrt{32}^{1/4}\left[\cos\left(\frac{\frac{5\pi}{4}+2(1)\pi}{4}\right)+i\sin\left(\frac{\frac{5\pi}{4}+2(1)\pi}{4}\right)\right]$$

$$\boxed{-1.282+0.857i}$$

★ $n=2$

$$\sqrt{32}^{1/4}\left[\cos\left(\frac{\frac{5\pi}{4}+2(2)\pi}{4}\right)+i\sin\left(\frac{\frac{5\pi}{4}+2(2)\pi}{4}\right)\right]$$

$$\boxed{-0.857-1.282i}$$

★ $n=3$

$$\sqrt{32}^{1/4}\left[\cos\left(\frac{\frac{5\pi}{4}+2(3)\pi}{4}\right)+i\sin\left(\frac{\frac{5\pi}{4}+(2)(3)\pi}{4}\right)\right]$$

$$\boxed{1.282-0.857i}$$