Secondary 3H: 6-2 Notes: Properties of Exponentials and Modeling Exponentials

Warm-Up: $y = a \cdot b^x$ b

- The value of an industrial machine has a decay factor of 0.75 per year. After six years, the machine is worth \$7500. What was the original value of the machine? a

x	y
0	a
6	7500

$$7500 = a(0.75)^6$$

$$a = \frac{7500}{(0.75)^6} = \boxed{\$42{,}139.92}$$

Modeling with Exponential Functions

- **Exponential Growth and Decay**
  - For exponential growth $y = ab^x$, with $b > 1$, the value b is the growth factor.
  - For exponential decay, where $0 < b < 1$, the value b is the decay factor.
  - A quantity that exhibits exponential growth increases by a constant percentage each time period. For exponential decay, the quantity decreases by a constant percentage each time period.
  - Give me some examples of things that increase or decrease by a constant percentage?

    bank interest, bacteria, half-life, car, bouncy ball, speed (?) in space
  - How can we use this constant percentage to help us find b?

    change percentage to decimal.
  - The *rate of increase* or *growth rate* is represented by r.
  - $\boxed{b = 1 + r}$ $y = ab^x \Rightarrow y = a(1+r)^x$
  - WHY?
  - Example: The cost of Christmas trees increases by 5% every year. Let's say the cost of a Christmas tree this year is \$40. How much would that same tree cost next year? $\boxed{\$42}$ r

    40 + increase

    $40 + 40(0.05)$

    $\underbrace{40}_{a}\underbrace{(1+0.05)}_{b}$
  - The *rate of decay* is also r. Usually a rate of decay is expressed as a negative quantity, so $b = 1 + r$ is still true.

    $b = 1 + (-r)$

    OR $b = 1 - r$

Exponential Growth and Decay

$$A(t) = \overset{\text{initial value}}{a}(1 + \overset{\text{rate}}{r})^{t} \leftarrow \text{time}$$

- To be exponential, a quantity must change by a fixed percentage each time period.

Example

- You invested $1000 (a) in a savings account at the end of the 6th grade. The account pays 5% (r) annual interest. How much money will be in the account after $\underset{t}{\text{six years}}$?

$y = 1000(1+0.05)^6$

$\boxed{y = \$1{,}340.10}$

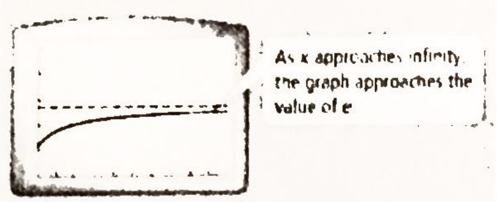
Your Turn

- Suppose you invest $500 (a) in a savings account that pays 3.5% (r) annual interest. How much will be in the account after $\underset{t}{\text{five}}$ years?

$y = 500(1+0.035)^5$

$\boxed{y = \$593.84}$

- **What if you compound interest more often than once a year?**
  - $y = \left(1 + \frac{1}{x}\right)^x$



  - Exponential functions with base e are useful for describing continuous growth or decay. These are called natural base exponential functions.

- **Using a Calculator to evaluate e^x**
  - $e^3 = 20.1$
  - $e^{1.5} = 4.5$
  - $e^{-5} = 0.007$

Continuously Compounded Interest

- The formula for continuously compounded interest uses the number e.

$$y = \underbrace{P}_{\text{principle}}\,\underbrace{e}_{\text{base \#}}^{\overbrace{r}^{\text{rate}}\overbrace{t}^{\text{time}}}$$

Let's try a few examples

- Suppose you won a contest at the start of 5^{th} grade that deposited $\boxed{\$3000}$ in an account that pays 5% annual interest compounded continuously. How much will you have in the account when you enter high school 4 years later?

a p r t

$y = Pe^{rt}$

$y = 3000e^{(0.05 \cdot 4)}$

$\boxed{y = \$3{,}664.21}$

- Using the same scenario, how much will be in the account after 4 years of high school?

$y = 3000e^{(0.05 \cdot 8)}$ OR $y = 3664.21e^{(0.05 \cdot 4)}$

$\boxed{y = \$4{,}475.47}$

- How long will it take until you have \$9000 in your bank account?

t y

$$\frac{\cancel{9000}}{\cancel{3000}} = \frac{\cancel{3000}e^{0.05t}}{\cancel{3000}}$$

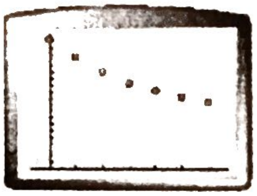
$3 = e^{0.05t}$

$\boxed{t \approx 22 \text{ years}}$

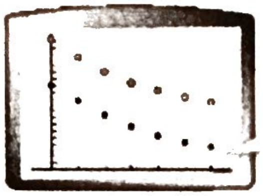
- <u>What about a different type of scenario?</u>
  - The best temperature to brew coffee is between 195°F and 205°F. Coffee is cool enough to drink at 185°F. The table shows temperature readings from a sample cup of coffee. How long does it take for a cup of coffee to be cool enough to drink?

Time (min)	Temp (°F)
0	203
5	177
10	153
15	137
20	121
25	111
30	104

Step 1
Plot the data to determine if an exponential model is realistic.

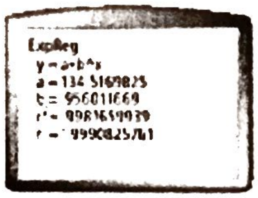


Step 2
The graphing calculator exponential model assumes the asymptote is $y = 0$. Since room temperature is about 68°F, subtract 68 from each temperature value. Calculate the third list by letting **L3 = L2 − 68**.

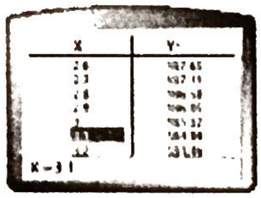


The graphing calculator exponential model assumes the asymptote is $y = 0$.

Step 3
Use the **ExpReg L1, L3** function on the transformed data to find an exponential model.



Step 4
Translate $y = 134.5(0.956)^x$ vertically by 68 units to model the original data. Use the model $y = 134.5 \cdot 0.956^x + 68$ to find how long it takes the coffee to cool to 185°F.



The coffee takes about 3.1 min to cool to 185°F.

- Use the model $y = 134.5 \cdot 0.956^x + 68$ from the last problem. How long does it take for the coffee to reach a temperature of 100 degrees?

x y

$100 = 134.5(0.956)^x + 68$

≈ 32 mins

A pot of water is heated to 200°F. The table shows typical temperature readings for the pot. The room temperature is 70°F. How long will it take the water to cool to 150°F?

x y

Time (min)	Temp (°F)
0	200
5	164
10	140
15	124
20	108
25	98

$y = ab^x$

$y = 129.54(0.941)^x + 70$

$150 = 129.54(0.941)^x + 70$