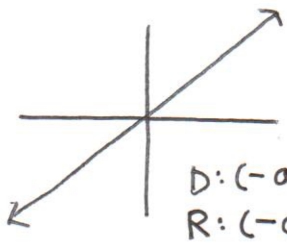
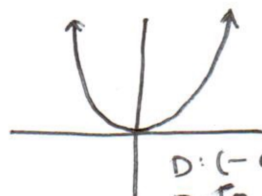
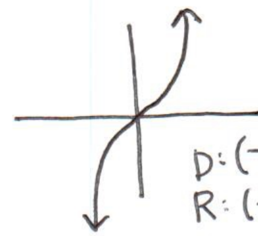
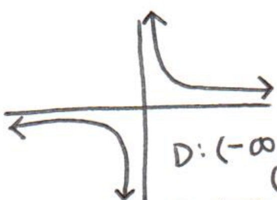
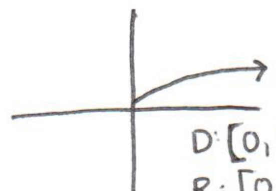
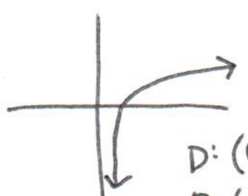
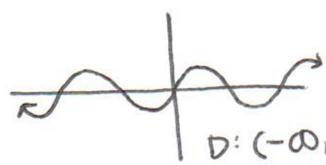
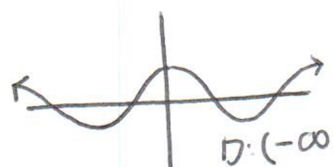
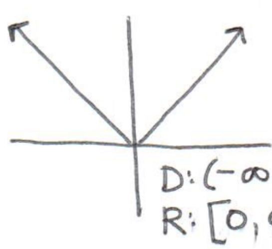
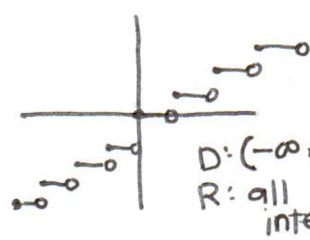
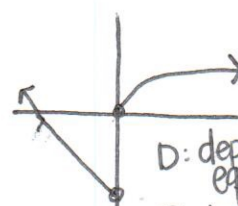
Secondary Math 3 Honors Unit 10: Functions Name: Key

Parent Functions

As you continue to study mathematics, you will find that the following functions will come up again and again. Please use the following table to paste your functions cut out from the additional page you received from your teacher. Make sure you match the correct graph with the correct equation and information.

Equation: $f(x) = x$  D: $(-\infty, \infty)$ R: $(-\infty, \infty)$ This is the only function that acts on every real number by leaving it alone.	Equation: $f(x) = x^2$  D: $(-\infty, \infty)$ R: $[0, \infty)$ The graph of this function, called a parabola, has a reflection property that is useful in making flashlights and satellite dishes.	Equation: $f(x) = x^3$  D: $(-\infty, \infty)$ R: $(-\infty, \infty)$ The origin is called a "point of inflection" for this curve because the graph changes curvature at that point.
Equation: $f(x) = \frac{1}{x}$  D: $(-\infty, 0) \cup (0, \infty)$ R: $(-\infty, 0) \cup (0, \infty)$ This curve, called a hyperbola, also has a reflection property that is useful in satellite dishes.	Equation: $f(x) = \sqrt{x}$  D: $[0, \infty)$ R: $[0, \infty)$ Put any positive number into your calculator. Take the square root. Then take the square root again. Then take the square root again, and so on. Eventually you will always get 1.	Equation: $f(x) = e^x$  D: $(-\infty, \infty)$ R: $(0, \infty)$ The number e is an irrational number (like π) that shows up in a variety of applications. This symbol was brought into popular use by the great Swiss mathematician Leonhard Euler.
Equation: $f(x) = \ln x$  D: $(0, \infty)$ R: $(-\infty, \infty)$ This function increases very slowly. If the x and y axis were scaled to 1 inch lengths, you would have to travel more than 2.5 miles along the curve just to get 1 foot above the x-axis.	Equation: $f(x) = \sin x$  D: $(-\infty, \infty)$ R: $[-1, 1]$ This function and the sinus cavities in your head derive their names from a common root; the Latin word for "bay".	Equation: $f(x) = \cos x$  D: $(-\infty, \infty)$ R: $[-1, 1]$ The local extrema of this function occur exactly at the zeros of the sine function and vice versa.
Equation: $f(x) = \lvert x \rvert$  D: $(-\infty, \infty)$ R: $[0, \infty)$ This function has an abrupt change of direction at the origin, while other functions are all "smooth" on their domains.	Equation: $f(x) = int(x)$  D: $(-\infty, \infty)$ R: all integers This function has a jump discontinuity at every integer value of x. Similar looking functions are called step functions.	Equation: Piecewise functions  D: depends on equation R: depends on equation These types of functions have more real-world examples because they model things where a rule changes suddenly.

Transformations

By acting on the x-coordinates and y-coordinates of points, transformations change graphs in predictable ways. Let's explore some of those changes.

1. Set your viewing window to $[-5,5]$ by $[-5,15]$ and your graphing mode to sequential as opposed to simultaneous and graph the following functions on the same screen.

$$y_1 = x^2$$
$$y_2 = y_1 + 3 = x^2 + 3$$
$$y_3 = y_1 + 1 = x^2 + 1$$
$$y_4 = y_1 - 2 = x^2 - 2$$
$$y_5 = y_1 - 4 = x^2 - 4$$

What effect do the +3, +1, -2, and -4 seem to have?

Move up or down, the number
(+) (−) tells you by how much.

2. Graph the functions on the same screen.

$$y_1 = x^2$$
$$y_2 = y_1(x+3) = (x+3)^2$$
$$y_3 = y_1(x+1) = (x+1)^2$$
$$y_4 = y_1(x-2) = (x-2)^2$$
$$y_5 = y_1(x-4) = (x-4)^2$$

What effect do the +3, +1, -2, and -4 seem to have?

Move left or right, The number tells you by how
(+) (−) much.

Repeat this process for the functions $y_1 = x^3, y_1 = |x|, y_1 = \sqrt{x}$. Do your observations agree with those you made after steps 1 and 2?

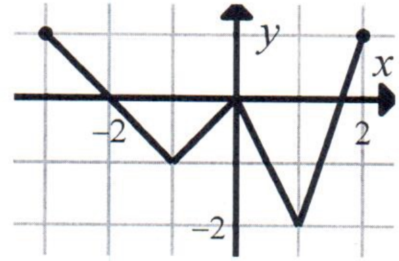
yes

Transformations may be performed in succession—one after another. If the transformations include stretches, shrinks, or reflections, the order in which the transformations are performed may make a difference. In those cases, be sure to pay particular attention to order. The following tell us the order in which transformations should occur.

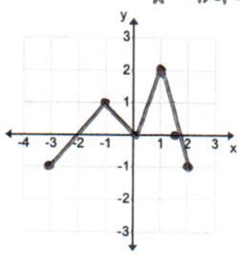
Graphing Adjustments to $y = f(x)$

1. $y = -f(x)$	reflect across the x-axis
2. $y = f(-x)$	reflect across the y-axis
3. $y = f(x) + d$	shift up if $d > 0$, shift down if $d < 0$
4. $y = f(x + c)$	shift left if $c > 0$, shift right if $c < 0$
5. $y = a \cdot f(x)$	vertical stretch if $a > 1$, vertical squeeze if $a < 1$ (assumes a is positive, if a is negative a reflection is needed)
6. $y = f(b \cdot x)$	horizontal squeeze if $b > 1$, horizontal stretch if $b < 1$ (assumes b is positive, if b is negative a reflection is needed)

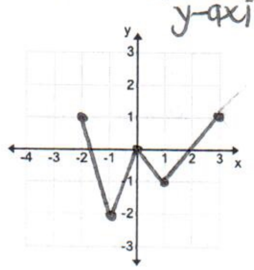
Use the graph of $y = f(x)$ to draw **accurate** graphs of the following.



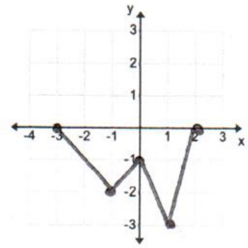
$y = -f(x)$ reflect across x-axis



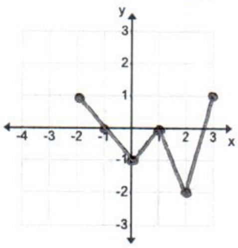
$y = f(-x)$ reflect across y-axis



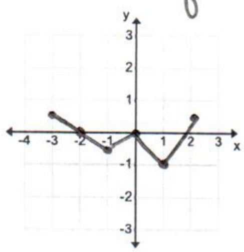
$y = f(x) - 1$ down 1



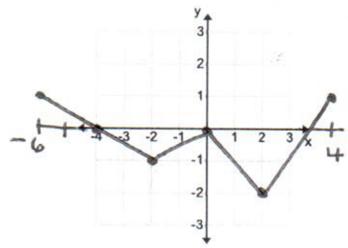
$y = f(x-1)$ Right 1



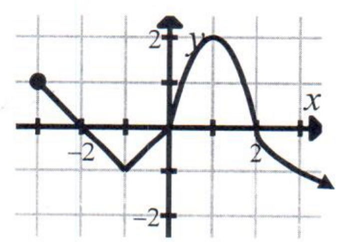
$y = \frac{1}{2} f(x)$ vertical squeeze



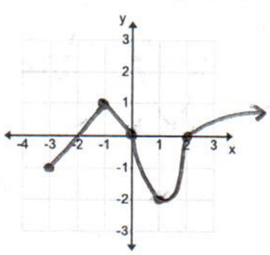
$y = f\left(\frac{1}{2}x\right)$ Horizontal stretch



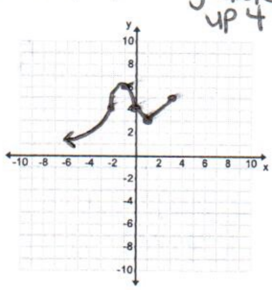
Use the graph of $y = f(x)$ to draw **accurate** graphs of the following.



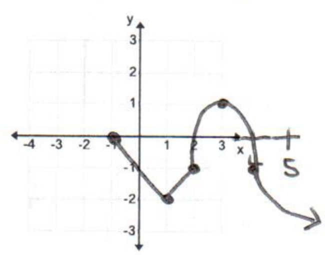
$y = -f(x)$ reflect across x-axis



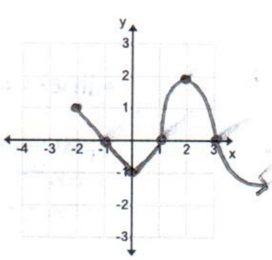
$y = f(-x) + 4$ reflect across y-axis up 4



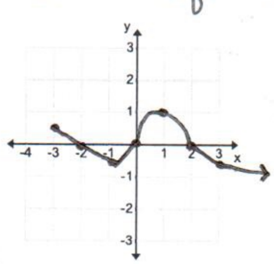
$y = f(x-2) - 1$ Right 2 down 1



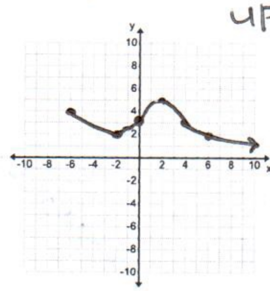
$y = f(x-1)$ Right 1



$y = \frac{1}{2} f(x)$ vertical squeeze



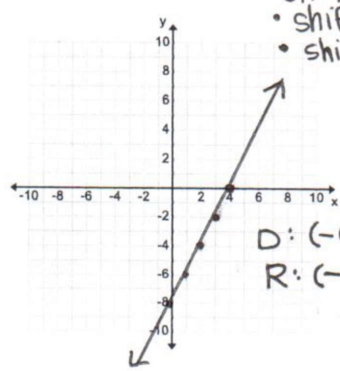
$y = f\left(\frac{1}{2}x\right) + 3$ Horizontal stretch up 3



Graph the following equations using what you know about the parent graphs and transformations. Then state domain and range.

$y = 2(x-3) - 2$

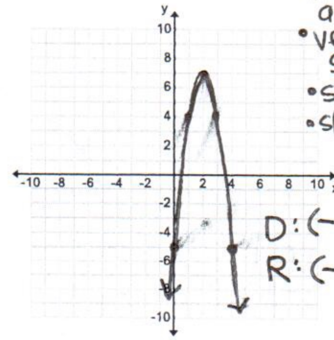
• vertical stretch
• shift Right 3
• shift down 2



D: $(-\infty, \infty)$
R: $(-\infty, \infty)$

$f(x) = -3(x-2)^2 + 7$

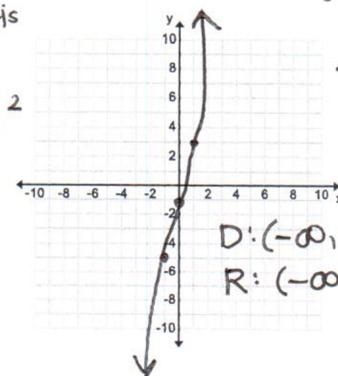
• Reflect across x-axis
• vertical stretch
• shift Right 2
• shift up 7



D: $(-\infty, \infty)$
R: $(-\infty, 7]$

$y = 4x^3 - 1$

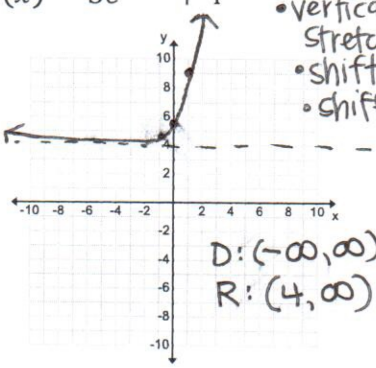
• vertical stretch
• down 1



D: $(-\infty, \infty)$
R: $(-\infty, \infty)$

$f(x) = 5e^{x-1} + 4$

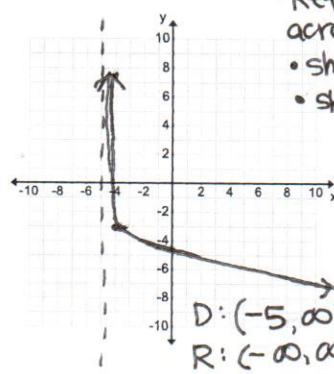
• vertical stretch
• shift left 1
• shift up 4



D: $(-\infty, \infty)$
R: $(4, \infty)$

$y = -\ln(x+5) - 3$

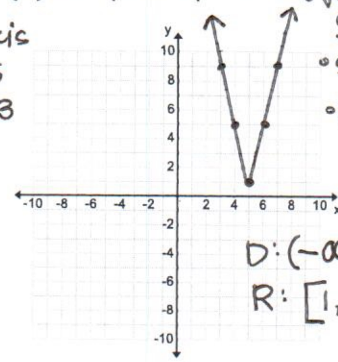
• Reflect across x-axis
• shift Left 5
• shift down 3



D: $(-5, \infty)$
R: $(-\infty, \infty)$

$f(x) = 4|x-5| + 1$

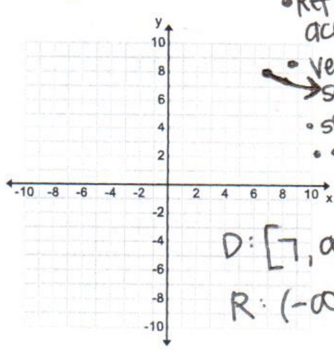
• vertical stretch
• shift right 5
• shift up 1



D: $(-\infty, \infty)$
R: $[1, \infty)$

$y = -\frac{1}{2}\sqrt{x-7} + 8$

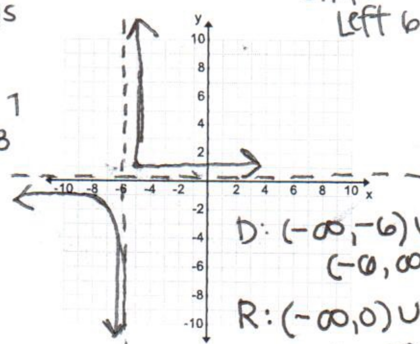
• Reflect across x-axis
• vertical squeeze
• shift Right 7
• shift up 8



D: $[7, \infty)$
R: $(-\infty, 8]$

$f(x) = \frac{1}{x+6}$

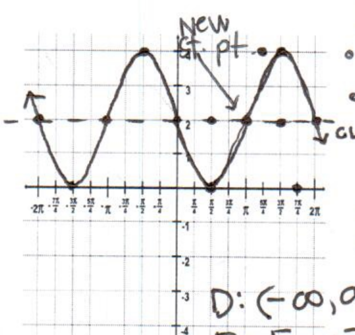
• shift Left 6



D: $(-\infty, -6) \cup (-6, \infty)$
R: $(-\infty, 0) \cup (0, \infty)$

$y = 2\sin(x - \pi) + 2$

• vertical stretch
• shift R π
• shift up 2

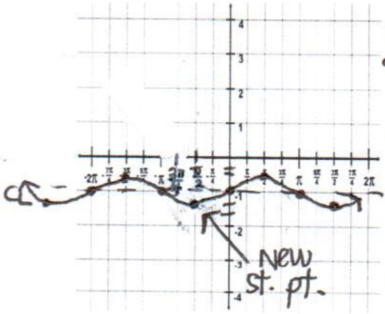


per: 2π
amp: 2

D: $(-\infty, \infty)$
R: $[0, 4]$

$f(x) = -\frac{1}{3}\cos\left(x + \frac{\pi}{2}\right) - 1$

• Reflect across x-axis
• vertical squeeze
• shift L $\frac{\pi}{2}$
• shift down 1

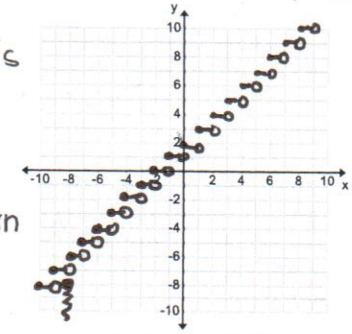


• per: 2π
• amp: $1/3$

D: $(-\infty, \infty)$
R: $\left[-\frac{4}{3}, -\frac{2}{3}\right]$

$y = int(x) + 2$

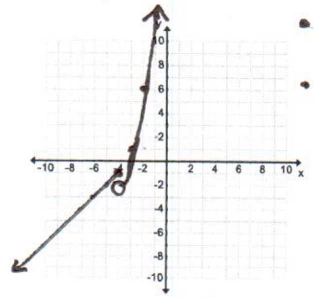
• up 2



D: $(-\infty, \infty)$
R: all integers

$f(x) = \begin{cases} x+3 & x \le -4 \\ (x+5)^2 - 3 & -4 < x \le 5 \end{cases}$

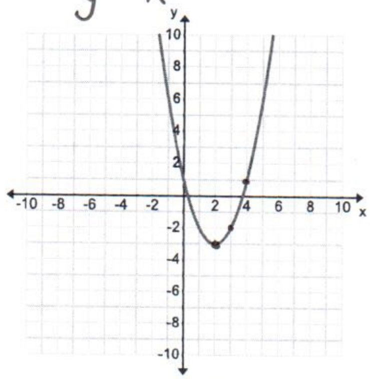
• line: shift up 3
• parabola: shift left 5, shift down 3



D: $(-\infty, \infty)$
R: $(-\infty, \infty)$

Given the graph, and what you know about parent functions and transformations, describe the transformations that occur then write the equations for the following graphs. Then state domain and range.

$y = x^2$

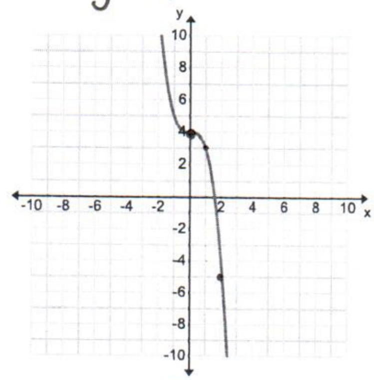


$y = (x-2)^2 - 3$

D: $(-\infty, \infty)$

R: $[-3, \infty)$

$y = x^3$

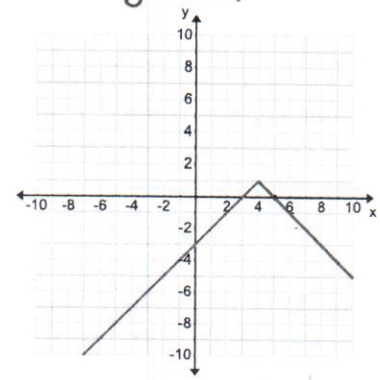


$y = -x^3 + 4$

D: $(-\infty, \infty)$

R: $(-\infty, \infty)$

$y = |x|$

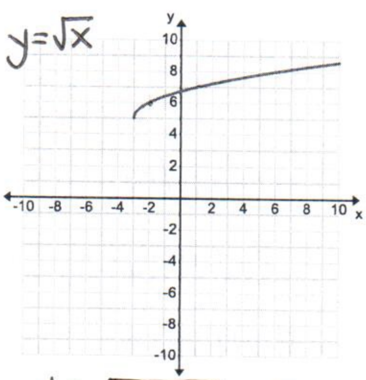


$y = -|x-4| + 1$

D: $(-\infty, \infty)$

R: $(-\infty, 1]$

$y = \sqrt{x}$

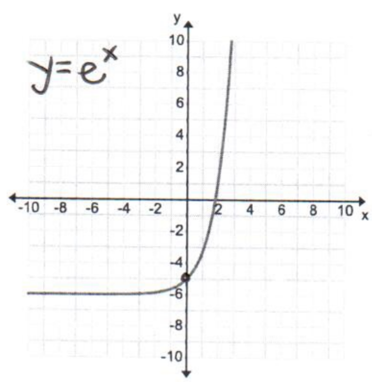


$y = \sqrt{x+3} + 5$

D: $[-3, \infty)$

R: $[5, \infty)$

$y = e^x$

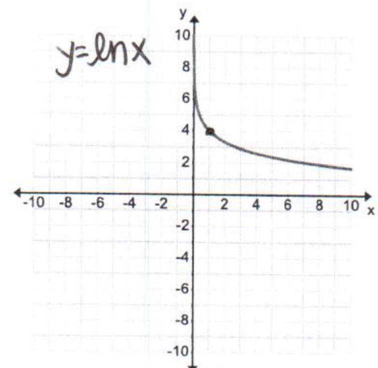


$y = e^x - 6$

D: $(-\infty, \infty)$

R: $(-6, \infty)$

$y = \ln x$

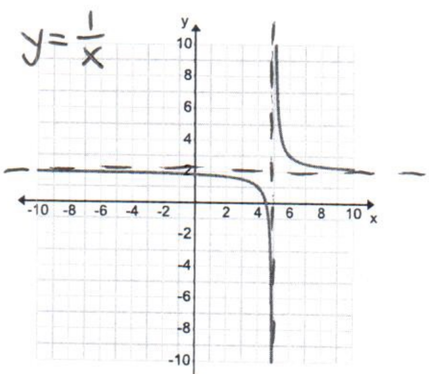


$y = -\ln(x) + 4$

D: $(0, \infty)$

R: $(-\infty, \infty)$

$y = \frac{1}{x}$

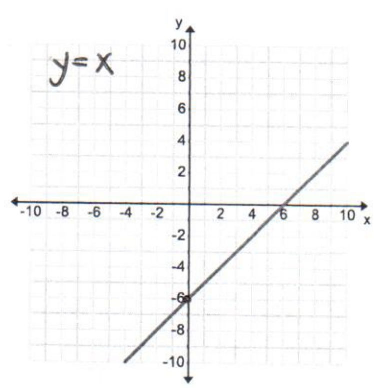


$y = \frac{1}{x-5} + 2$

D: $(-\infty, 5) \cup (5, \infty)$

R: $(-\infty, 2) \cup (2, \infty)$

$y = x$

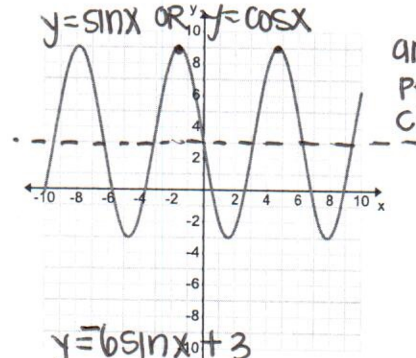


$y = x - 6$ OR $y = (x-6)$

D: $(-\infty, \infty)$

R: $(-\infty, \infty)$

$y = \sin x$ OR $y = \cos x$



amp: 6

per: 2π

CL

$y = 6\sin x + 3$

OR

$y = 6\sin(x+\pi) + 3$

OR

$y = 6\cos(x + \frac{\pi}{2}) + 3$

OR

$y = -6\cos(x - \frac{\pi}{2}) + 3$

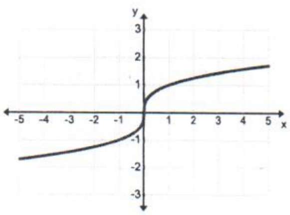
D: $(-\infty, \infty)$

R: $[-3, 9]$

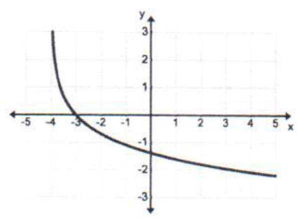
Increasing/Decreasing

- A function f is **increasing** on an interval if, for any two points in the interval, a positive change in x results in a positive change in $f(x)$.
- A function f is **decreasing** on an interval if, for any two points in the interval, a positive change in x results in a negative change in $f(x)$.
- A function f is **constant** on an interval if, for any two points in the interval, a positive change in x results in a zero change in $f(x)$.

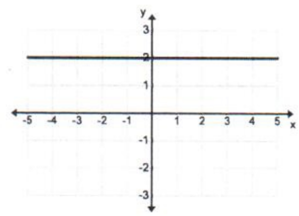
The following graphs show examples of increasing, decreasing, or constant on an interval.



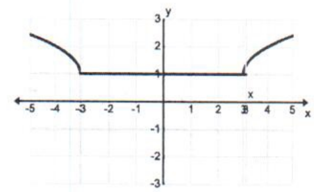
Increasing $(-\infty, \infty)$



Decreasing $(-\infty, \infty)$



Constant $(-\infty, \infty)$

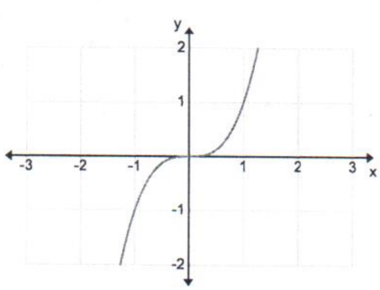


Decreasing on $(-\infty, -3]$
Constant on $[-3, 3]$
Increasing on $[-3, \infty)$

- For increasing functions when moving from *left* to *right*, the graph has a positive slope.
- For decreasing functions when moving from *left* to *right*, the graph has a negative slope.
- For constant functions wen moving from *left* to *right*, the graph has constant slope.

Determine the open interval on which each function is increasing, decreasing, or constant. Write your answers in interval notation.

$f(x) = x^3$

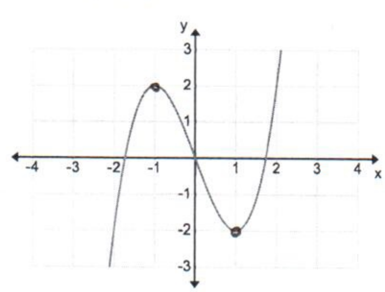


Increasing: $(-\infty, \infty)$

Decreasing: None

Constant: None

$f(x) = x^3 - 3x$

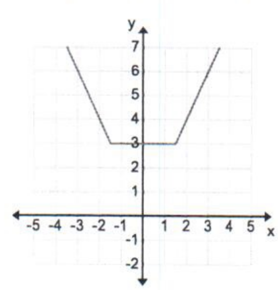


Increasing: $(-\infty, -1], [1, \infty)$

Decreasing: $[-1, 1]$

Constant: NONE

$f(x) = |x - 1.5| + |x + 1.5|$



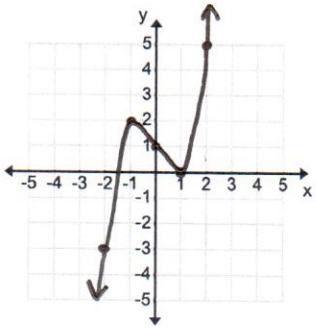
Increasing: $[1.5, \infty)$

Decreasing: $(-\infty, -1.5]$

Constant: $[-1.5, 1.5]$

Graph the function in your calculator and provide a sketch of the graph. Then determine the open intervals on which each function is increasing and decreasing.

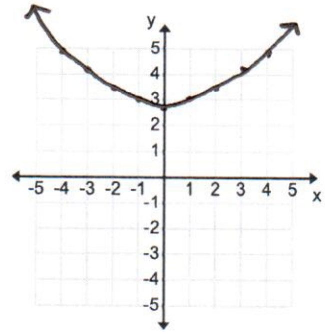
$f(x) = x^3 - 2x + 1$



incr: $(-\infty, -1]$ $[1, \infty)$

decr: $[-1, 1]$

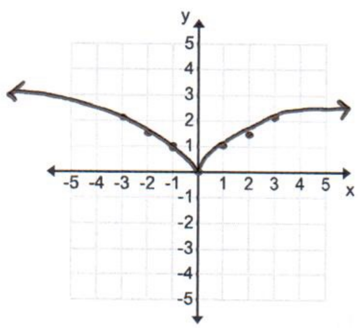
$f(x) = \sqrt{x^2 + 8}$



incr: $[0, \infty)$

decr: $(-\infty, 0]$

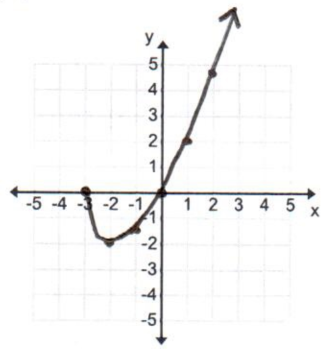
$f(x) = x^{\frac{2}{3}}$



incr: $[0, \infty)$

decr: $(-\infty, 0]$

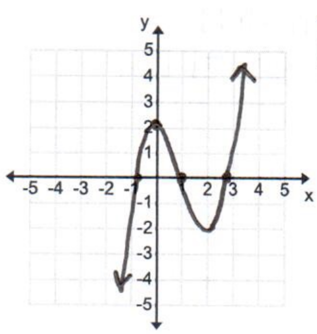
$f(x) = x\sqrt{x + 3}$



incr: $[-2, \infty)$

decr: $[-3, -2]$

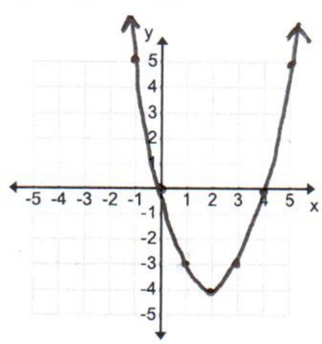
$f(x) = x^3 - 3x^2 + 2$



incr: $(-\infty, 0]$ $[2, \infty)$

decr: $[0, 2]$

$f(x) = x^2 - 4x$

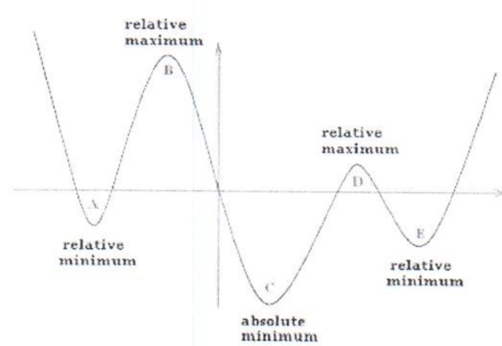


incr: $[2, \infty)$

decr: $(-\infty, 2]$

Extrema

Many graphs are characterized by peaks and valleys where they change from increasing to decreasing and vice versa. The extreme values of a function (or the local extrema) can be characterized as either *local (relative) maxima* or *local (relative) minima*. The distinction can be easily seen graphically at the right.



- Relative maximums occur when the function changes from increasing to decreasing.
- Relative minimums occur when the function changes from decreasing to increasing.

Use your calculator and the steps below to find the exact relative minimum and relative maximum of the function given by $f(x) = -x^3 + x$.

1. Graph the function in y_1
2. Adjust your window so you can see all the relative extrema.
3. Press the "Calc" button by pressing "2nd trace"
4. Choose "minimum"
5. Your calculator is now going to ask for a left bound—scroll with the arrow keys so that you are on the left side of one of the extrema and press enter.
6. Your calculator is now going to ask for a right bound—scroll with the arrow keys so that you are on the right side of the same extrema and press enter.
7. Your calculator is now going to ask for a guess—scroll with the arrow keys so that you are really close to the extrema and press enter.
8. Your calculator now will tell you x and y values of the extrema.
   * Note: If the question asks for the value or the relative minimum they are only looking for the y-value. If the question asks for the place where the extrema occur they are looking for the x-value. If the question asks for the point where the extrema occur they are looking for both the x- and y-value.
9. Now repeat these steps to find any other extrema on the graph.

Use your calculator to find the exact point (to 3 decimal places) of any relative extrema of the following:

$f(x) = (x-1)\sqrt{x}$

Minimum
(0.333, −0.385)

$f(x) = 4$

NONe

$f(x) = x^3 - 6x^2 + 15$

Maximum
(0, 15)
Minimum
(4.000, −17)

$f(x) = x^2 + 5x + 9$

Minimum
(−2.5, 2.75)

$f(x) = x\sqrt{4-x}$

Maximum
(2.667, 3.079)

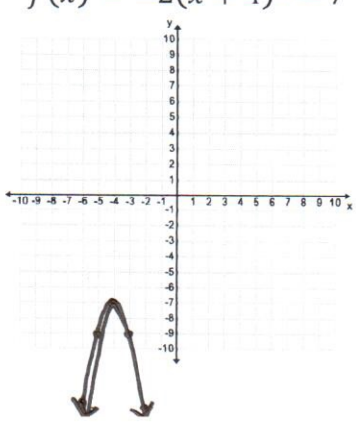
$f(x) = 7x + 12$

NONe

Use what you have learned so far about functions to complete the following.

Graph the following function ***without*** a calculator:

$f(x) = -2(x+4)^2 - 7$



Find each of the following:

Domain: $(-\infty, \infty)$

Range: $(-\infty, -7]$

Increasing: $(-\infty, -4)$

Decreasing: $(-4, \infty)$

Relative maximum: $(-4, -7)$

Relative minimum: NONe

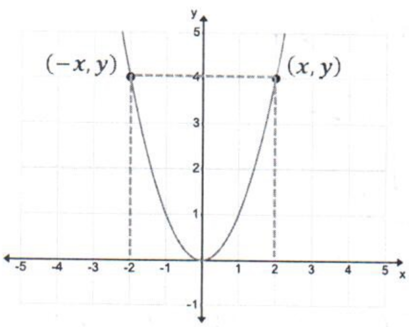
Symmetry

In the graphical sense, the word "symmetry" in mathematics carries essentially the same meaning as it does in art: The picture (in this case, the graph) "looks the same" when viewed in more than one way. The interesting thing about mathematical symmetry is that it can be characterized numerically and algebraically as well. We will be looking at two particular types of symmetry, each of which can be spotted easily from a graph, a table of values, or an algebraic formula, once you know what to look for.

Symmetry with respect to the y-axis (EVEN FUNCTIONS)

Example: $f(x) = x^2$

Graphically



The graph looks the same to the left of the y-axis as it does the right of it.

Numerically

x	$f(x)$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

Algebraically

For all x in the domain of f,
$f(-x) = f(x)$.

Functions with this property are "even" functions.

$$f(x) = x^2$$
$$f(-x) = (-x)^2 = x^2$$

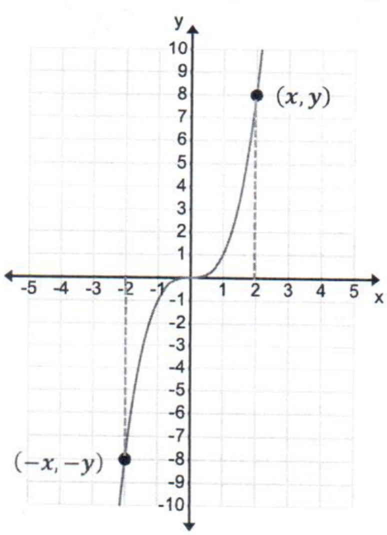
When $f(-x)$ and $f(x)$ are **E**xactly the same the function is **E**ven.

Exactly the same → Even

Symmetry with respect to the origin (ODD FUNCTIONS)

Example: $f(x) = x^3$

Graphically



The graph looks the same upside down as it does right side up.

Numerically

x	$f(x)$
-3	-27
-2	-8
-1	1
0	0
1	1
2	8
3	27

Algebraically

For all x in the domain of f,
$f(-x) = -f(x)$.

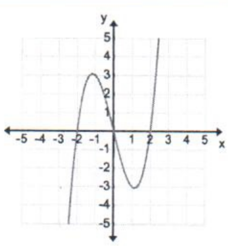
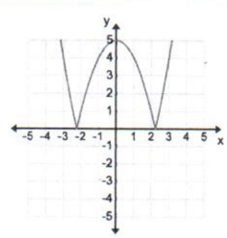
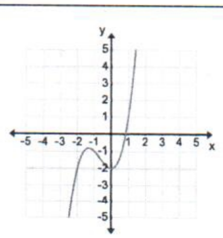
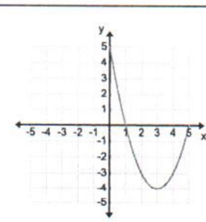
Functions with this property are "odd" functions.

$$f(x) = x^3$$
$$f(-x) = (-x)^3 = -x^3$$

When $f(-x)$ and $f(x)$ are **O**pposite in sight then the function is **O**dd.

Opposite →Odd

Tell whether each of the following functions is odd, even, or neither. The first two are done for you.

$f(x) = x^2 - 3$ $f(-x) = (-x)^2 - 3 = x^2 - 3 = f(x)$ $f(-x) = f(x)$ The function is even!	$g(x) = \dfrac{x^3}{4 - x^2}$ $g(-x) = \dfrac{(-x)^3}{4-(-x)^2} = -\dfrac{x^3}{4-x^2} = -g(x)$ $g(-x) = -g(x)$ The function is odd!
$f(x) = x^2 - 2x - 2$ $f(-x) = (-x)^2 - 2(-x) - 2$ $= x^2 + 2x - 2$ $f(-x) \neq f(x)$ $f(-x) \neq -f(x)$ $\boxed{\text{Neither}}$	$f(x) = \sqrt{x^2 + 2}$ $f(-x) = \sqrt{(-x)^2 + 2}$ $= \sqrt{x^2 + 2}$ $\boxed{\text{EVEN}}$
$f(x) = 2x^3 - 3x$ $f(x) = 2(-x)^3 - 3(-x)$ $= -2x^3 + 3x$ $\boxed{\text{ODD}}$	$f(x) = x^3 + 0.04x^2 + 3$ $f(x) = (-x)^3 + 0.04(-x)^2 + 3$ $= -x^3 + 0.04x^2 + 3$ $\boxed{\text{NEITHER}}$
$f(x) = \dfrac{1}{x}$ $f(-x) = \dfrac{1}{-x}$ $\boxed{\text{ODD}}$	$f(x) = \dfrac{x^3}{5 - x^3}$ $f(-x) = \dfrac{(-x)^3}{5-(-x)^3} = \dfrac{-x^3}{5+x^3} = -\dfrac{x^3}{5+x^3}$ $\boxed{\text{Neither}}$
 $\boxed{\text{ODD}}$	 $\boxed{\text{EVEN}}$
 $\boxed{\text{NEITHER}}$	 $\boxed{\text{NEITHER}}$

Limits

A limit is telling us what is happening to a function as the x gets close to a specific value from *both sides*. If the function does not approach the same value from both sides the limit does not exist (DNE). Limits can be found 3 ways: from a table (numerically), by looking at a graph (graphically), and algebra (algebraically). We have specific notation that you need to use and get used to for limits. The following: $\lim_{x \to 4} f(x) = 9$ is read as "the limit of $f(x)$ as x approaches 4 is 9" or "the limit as x approaches 4 of $f(x)$ is 9."

Fill in the table to help you find the following: What is limit of $f(x) = 3x^2 + x + 1$ as x approaches 3?

Notation: $\lim_{x \to 3} f(x) =$ 31 (since the function approaches 31 from both sides of the table.)

x	2.5	2.6	2.7	2.8	2.9	3	3.1	3.2	3.3	3.4
$f(x)$	22.25	23.88	25.57	27.32	29.13	31	32.93	34.92	36.97	39.08

Use the tables to numerically estimate the limits.

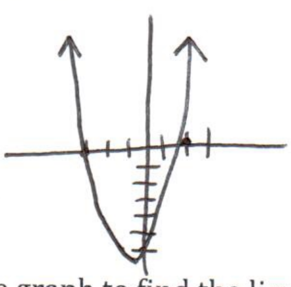
$\lim_{x \to 2}(3x - 2) = 4$

x	1.7	1.8	1.9	2	2.1	2.2	2.3	2.4
$f(x)$	3.1	3.4	3.7	4	4.3	4.6	4.9	5.2

$\lim_{x \to 0}\left(\frac{x}{\sqrt{x+1} - 1}\right) = 2$

x	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
$f(x)$	1.84	1.89	1.95	error	2.05	2.09	2.14	2.18

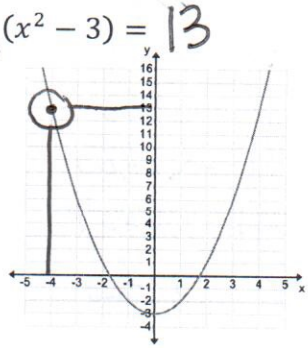
Sketch the graph the following and use the graph to determine the limit of $f(x) = x^2 + x - 6$ as x approaches 2.



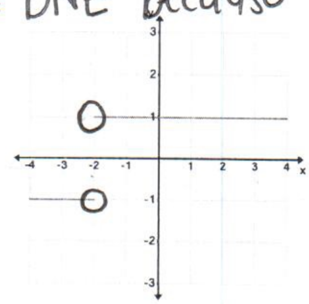
Notation: $\lim_{x \to 2} x^2 + x - 6 = 0$

Use the graph to find the limit (if it exists). If the limit does not exist, explain why not.

$\lim_{x \to -4}(x^2 - 3) = 13$



$\lim_{x \to -2} \frac{|x+2|}{x+2} =$ DNE because LHL and RHL are not the same.



But, what if the values on either side are different?

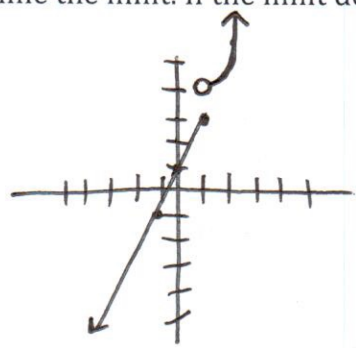
What is the limit of $f(x)=\begin{cases}2x+1 & \text{if } x\le 1\\ x^2+3 & \text{if } x>1\end{cases}$ as x approaches 1?

Sketch the graph the following and use the graph to determine the limit. If the limit does not exist, explain why not.

Notation:

$\lim_{x\to 1} f(x) = \text{DNE}$

The left hand side does not approach the right hand side.



LEFT-HAND/ RIGHT-HAND LIMITS: These are called one-sided limits and only look at the function as x approaches from one side-either the left or the right.

Left-Hand Limits: Coming from the left or from the negatives

Notation: $\lim_{x\to a^-} f(x) =$

Right-Hand Limits: Coming from the right or from the positives

Notation: $\lim_{x\to a^+} f(x) =$

Use this idea to help complete the previous example. Find the following limits:

- $\lim_{x\to 1^-} f(x) = 3$
- $\lim_{x\to 1^+} f(x) = 4$

Find the left-hand and right-hand limits of the function using tables.

$f(x)=x^2+3x-11$

LHL $\lim_{x\to 0^-} f(x)$

x	y
-1	-13
-0.5	-12.25
-0.3	-11.81
-0.1	-11.29
-0.001	-11.003
0	-11

RHL $\lim_{x\to 0^+} f(x)$

x	y
1	-7
0.7	-8.41
0.5	-9.25
0.1	-10.69
0.0001	-10.999
0	-11

$f(0)=0^2+3(0)-11=-11$

$f(x)=\dfrac{x^2+3x}{-4x-8}$

RHL $\lim_{x\to -2^-} f(x)$

x	y
-1	0.5
-1.5	1.125
-1.7	1.842
-1.9	5.225
-1.999	500.25
-2	error

LHL $\lim_{x\to -2^+} f(x)$

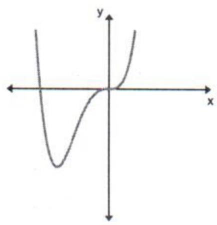
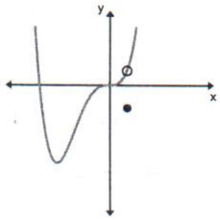
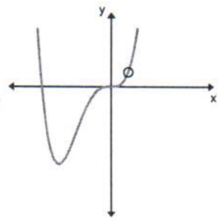
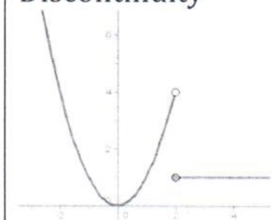
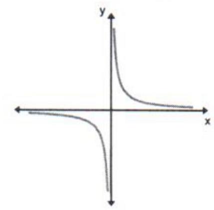
x	y
-3	0
-2.5	-0.625
-2.1	-4.725
-2.01	-49.748
-2.0001	-4999.75
-2	error

vertical asymptote

$\lim_{x\to -2^-} f(x) = -\infty$ $\lim_{x\to -2^+} f(x) = \infty$

Continuity

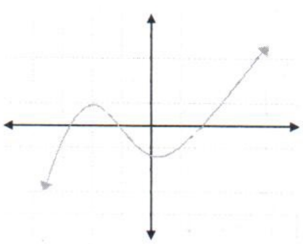
One of the most important properties of the majority of functions that model real-world behavior is that they are *continuous*. Graphically speaking, a function is continuous at a point if the graph does not come apart at that point. A function is discontinuous if the graph comes apart at any points. A good way to think about continuity is: can you trace the function without lifting your pencil? We can illustrate the concept with a few graphs below.

Continuous	Removable Discontinuity	Removable Discontinuity	Jump Discontinuity	Infinite Discontinuity
				
This graph is continuous everywhere. Notice that the graph has no breaks.	This graph is continuous everywhere except for the "hole". This is called a removable discontinuity because it can be patched by redefining the function at the point so as to plug the hole.	This graph also has a removable discontinuity.	Here is a discontinuity that is not removable. It is a jump discontinuity because there is more than just a hole; there is a jump in the function that makes the gap impossible to plug with a single point no matter how we try to redefine the function.	This is a function with an infinite discontinuity because it has a vertical asymptote that is impossible to remove.

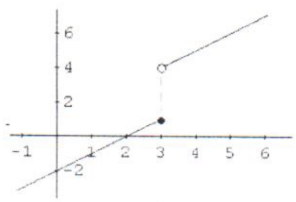
In order for a function to be considered continuous at a specific point, the following 3 steps must be met.

1. The function must exist at the point.
2. For any value of a function, the limit from the right and left hand side have to be equal.
3. When you plug the x-value into the function, it also needs to be equal to the value to the limit.

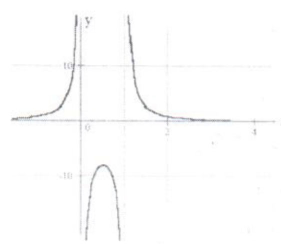
Determine if the following functions are continuous. If not state which type of discontinuity it has.



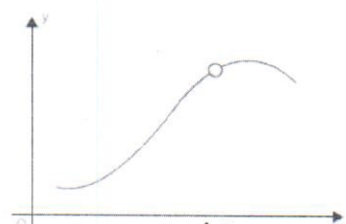
continuous



discontinuous
jump discontinuity



infinity
discontinuity



Removable
discontinuity
(hole)

Determine if $f(x) = \frac{3x+7}{x-2}$ is continuous at $x = 0$.

$\lim_{x \to 0^-} \frac{3x+7}{x-2} = -3.5$ $\quad$ $\lim_{x \to 0^+} \frac{3x+7}{x-2} = -3.5$

$f(0) = \frac{3(0)+7}{0-2} = -3.5$

$\boxed{\text{yes}}$

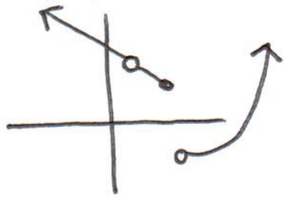
Are there any values of x that would make this function discontinuous? Why or why not?

x=2 asymptote

What type of functions are always continuous?

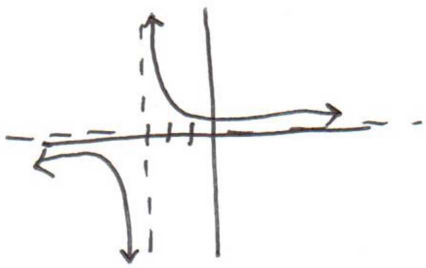
line, parabola, s-curve

What would the graph of a function with a jump and a hole look like?



Graph the following on your calculator and describe the discontinuities, if there are any.

$f(x) = \dfrac{2}{x+3}$



infinite discontinuity @ $x = -3$

Find the following limits:

$\lim_{x\to -3^+} f(x) = \infty$

$\lim_{x\to -3^-} f(x) = -\infty$

Without graphing, find the left-hand and right hand limits of the function using tables.

$f(x) = \dfrac{x-1}{x-2}$

$\lim_{x\to 2^-} f(x) =$				$\lim_{x\to 2^+} f(x) =$			
	X	Y			x	y	
	1.94	-15.667			2.01	101	∞
	1.95	-19			2.02	51	
	1.96	-24			2.03	34.333	
	1.97	-32.333			2.04	26	
	1.98	-49			2.05	21	
	1.99	-99			2.06	17.667	

$-\infty$

Using the tables, above describe any discontinuities for the function.

infinite discontinuity

What would a hole discontinuity look like using tables?

error for actual number

What would a jump discontinuity look like using tables?

approaches different values

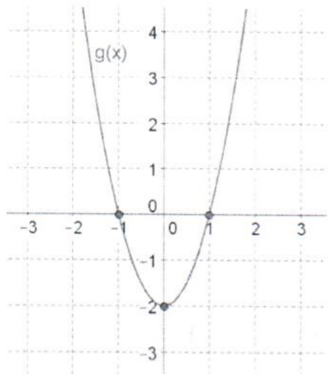
Comparing Functions

In this section, you will learn how to compare various characteristics of functions. We will look at functions in the form of a formula, a graph, a table of values, or within a context of a story. You should already know the following key terms: Roots, y-intercept, domain, range, intervals of increase/decrease, and rate of change.

Example 1: Given the two functions, answer the following questions.

Function #1: $f(x) = x^2 + x - 2$

Function #2: $g(x)$



1. What function has the smallest minimum value?

f(x): minimum (−0.5, −2.25)

g(x): minimum (0, −2)

so f(x)!

2. What function has the smallest root?

x-intercept

f(x): (−2, 0)

g(x): (−1, 0)

so f(x)!

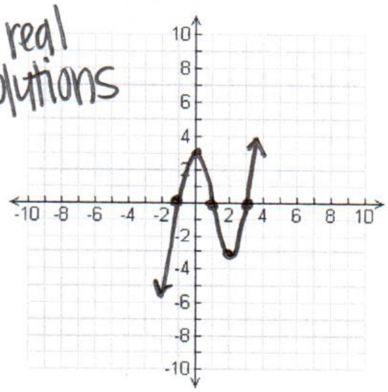
Example 2: Determine whether $f(x)$, a 3rd degree polynomial with values given on the table below, or $g(x) = x^4 - 2x^3 - 11x^2 - 8x - 60$, has more real solutions.

x-intercepts

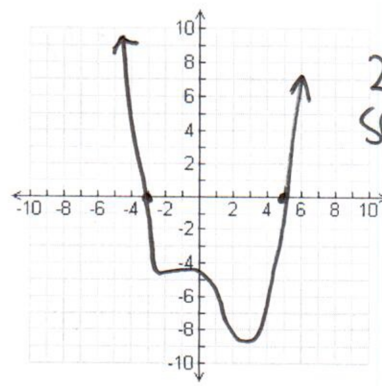
x	$f(x)$	
-1	0	★
0	3	
1	0	★
2	-3	
3	0	★

Graph of $f(x)$

3 real solutions



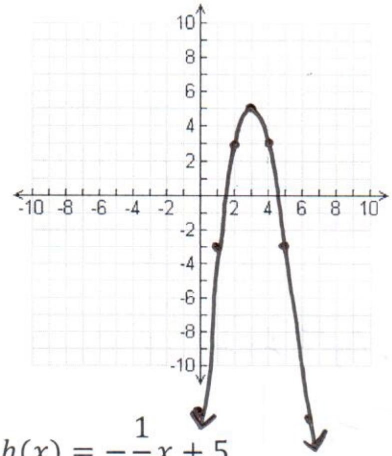
Graph of $g(x)$



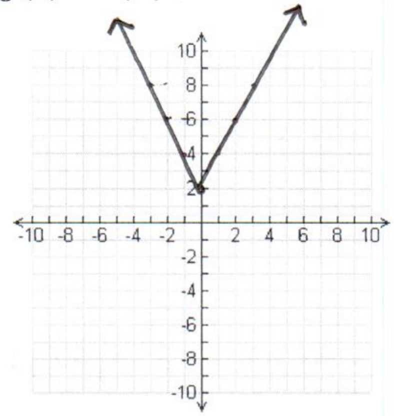
2 real solutions

Example 3: Graph each of the following functions and answer the questions below.

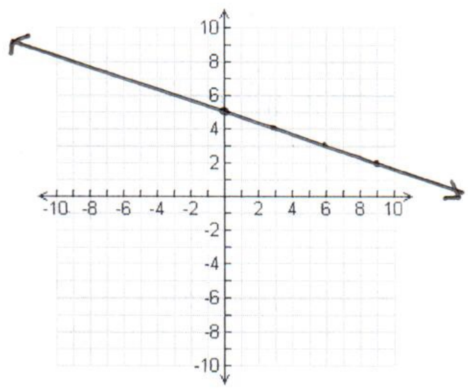
A. $f(x) = -2(x-3)^2 + 5$



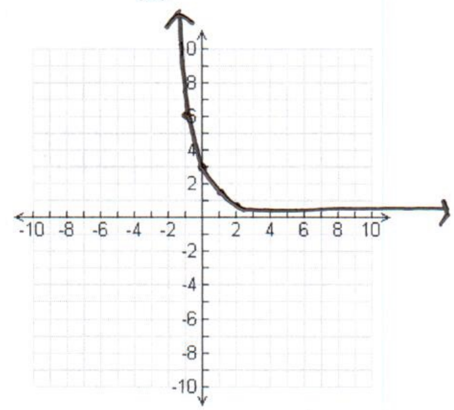
B. $g(x) = 2|x| + 2$



C. $h(x) = -\frac{1}{3}x + 5$

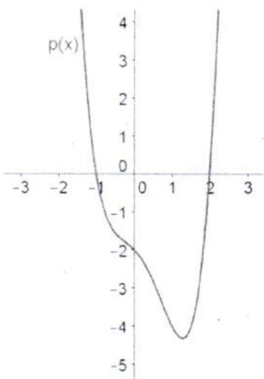


D. $k(x) = 3\left(\frac{1}{2}\right)^x$

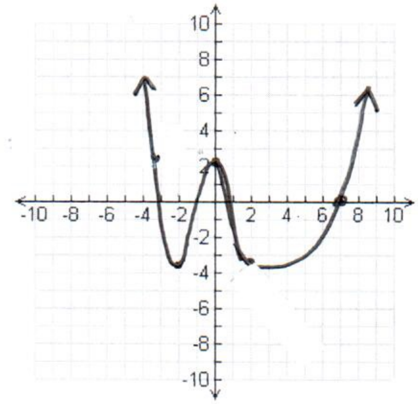


Compare the domain of each function. all have a domain of $(-\infty, \infty)$	Which function has the greatest range? $h(x)$ R: $(-\infty, \infty)$	Compare the intervals of increase of each graph. $f(x)$: $(-\infty, 3]$ $g(x)$: $[0, \infty)$ $h(x)$: NONE $k(x)$: NONE
Compare the intervals of decrease of each graph $f(x)$: $[3, \infty)$ $g(x)$: $(-\infty, 0)$ $h(x)$: $(-\infty, \infty)$ $k(x)$: $(-\infty, \infty)$	Which function has the greatest relative maximum? $f(x)$ Max @ $(3,5)$	What real world scenarios could you model with these graphs? your own answers

1. $p(x)$, graphed below and $q(x)$, with values given on the table below, are 4th degree polynomials.



x	$q(x)$
−3.18	2.53
−2	−3.75
0	2.25
2	−3.75



a. How many roots does each function have? 2 real roots (p(x))
4 real roots (q(x))

b. Which function has more real roots?
q(x)

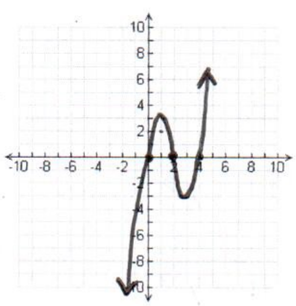
c. How many intervals of increase and decrease does $p(x)$ have?
1 decrease $(-\infty, 1]$
1 increase $[1, \infty)$

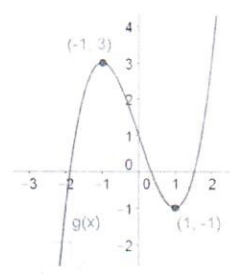
d. How many intervals of increase and decrease does $q(x)$ have?
2 increase
$(-2, 0)$ and $(2, \infty)$
2 decrease
$(-\infty, -2)$ and $(0, 2)$

e. If $q(x)$ has a known root of $x = 7$, does the end behavior of $q(x)$ match the end behavior of $p(x)$?
yes.

2. $f(x)$ and $g(x)$, graphed below, are both third degree polynomials.

$$f(x) = x^3 - 6x^2 + 8x$$





Both $(-1, 3)$ and $(1, -1)$ are points of relative extrema on $g(x)$.

a. Which function has a longer interval of decrease?
f(x): [1,3]
g(x): [-1,1] they are the same

b. Which function has a larger domain interval?
f(x): $(-\infty, \infty)$
g(x): $(-\infty, \infty)$ they are the same.

Composition

It is not hard to see that the function $\sin(x^2)$ is built from the basic functions $\sin x$ and x^2, but the functions are not put together by addition, subtraction, multiplication, or division. Instead, the two functions are combined by simply applying them in order—first the squaring function, then the sine function. This operation for combining functions is called function composition.

Definition: Composition of Functions: The composition f of g, denoted $f \circ g$, is defined by the rule $(f \circ g)(x) = f(g(x))$. This means we put $g(x)$ into $f(x)$. ORDER MATTERS HERE!!!

This next example will refresh your memory on function composition.

Let $f(x) = e^x$ and $g(x) = \sqrt{x}$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$ and verify numerically that they are not the same.

$$(f \circ g)(x) = f\big(g(x)\big) = f\big(\sqrt{x}\big) = e^{\sqrt{x}}$$
$$(g \circ f)(x) = g\big(f(x)\big) = g(e^x) = \sqrt{e^x}$$
$$e^{\sqrt{x}} \neq \sqrt{e^x}$$

Use the following functions to perform the following operations:

$f(x) = x^3$ $\quad g(x) = 2x + 7$ $\quad h(x) = x + 2$ $\quad j(x) = 4 - x^2$ $\quad k(x) = \sqrt{x^2 + 4}$ $\quad m(x) = x^3 + 1$

1. $f(g(x))$

$(2x+7)^3$

$= 8x^3 + 84x^2 + 294x + 343$

2. $g(f(x))$

$2(x^3)+7$

$= 2x^3 + 7$

3. $f(f(x)$

$(x^3)^3$

$= x^9$

4. $g(g(x))$

$2(2x+7)+7$

$= 4x+14+7$

$= 4x+21$

5. $(h \circ j)(x)$

$(4-x^2)+2$

$= -x^2+6$

6. $(j \circ h)(x)$

$4-(x+2)^2$

$= 4-x^2-4x-4$

$= -x^2-4x$

7. $(k \circ m)(x)$

$\sqrt{(x^3+1)^2+4}$

$= \sqrt{x^6+2x^3+1+4}$

$= \sqrt{x^6+2x^3+5}$

8. $(j \circ f)(x)$

$4-(x^3)^2$

$= 4-x^6$

Inverses

Problem to consider: Nathan received $100 as a birthday gift and has decided to save this money as well as add $20 of his own money each month.

a. Write an equation f(x) to model how much money Nathan will have in x months.

$f(x) = 100 + 20x$

b. How much money will Nathan have saved after 6 months?

$f(6) = 100 + 20(6) = \$220$

c. How long will it take Nathan to save $400?

$y = 400 \quad 400 = 100 + 20x \quad x = 15$ months

d. Write an equation g(x) to model how many months it will take Nathan to save x dollars.

$g(x) = \dfrac{x - 100}{20}$

Vocabulary:

- Definition of an inverse relation: The domain and range values switch

$x \leftrightarrow y$

Example 1: Below is a table of values. Fill in the table of values for the inverse

x	y
-3	-1
-1	1
9	11
23	25

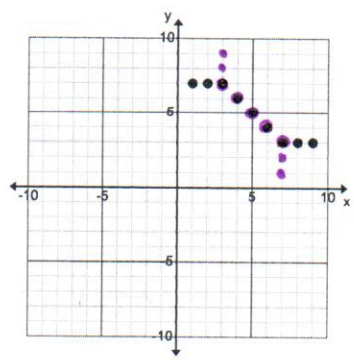
x	y
-1	-3
1	-1
11	9
25	23

Explain how you knew what values to put in the new table.

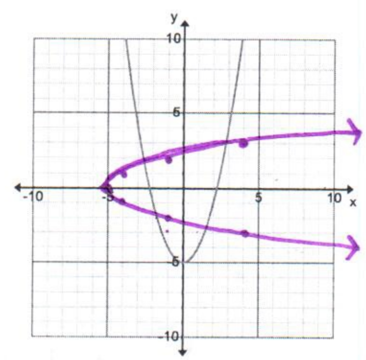
switch x and y.

Example 2: On the same grid, sketch the inverse of each given graph.

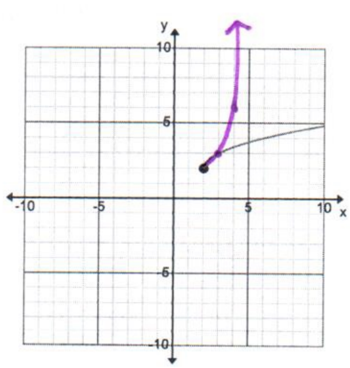
3.



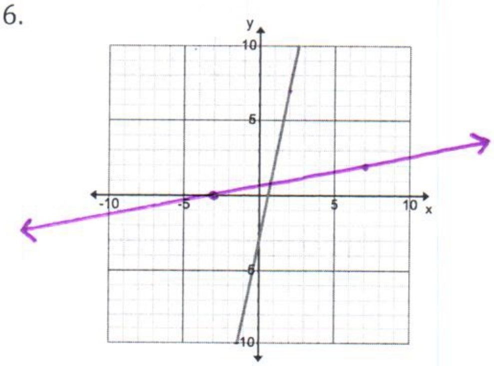
4.



5.



6.



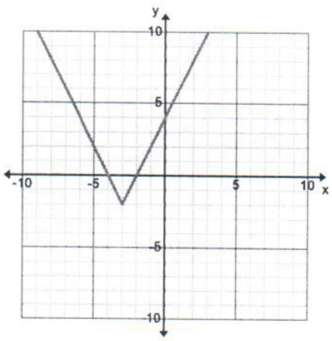
Vertical and Horizontal line test.

What is your conjecture about the relationship of the graph of a function and the graph of its inverse?

Because the x and y switch, the graph is reflected across $y=x$.

When we graphed the inverses on example 2, not all the graphs of the inverses were functions as well. How can you predict whether or not the original function's inverse will also be a function without actually graphing the inverse? does it pass the HLT?

Using your prediction, predict whether the following functions' inverse is also a function.



NO.

> Some functions are important enough that we want to study their inverse behavior even if they are not one-to-one (don't have an inverse).
>
> What could we do to make the function on the left have an inverse that we could look at? (Hint: Think about the graph of $\sin^{-1} x$)
>
> Restrict the domain.

Vertical line test: Indicates if the graph is a function (for every input, there is only one output).

Horizontal line test: Indicates if the function is one-to-one (for every output, there is only one input) and therefore tells us if the inverse is a function or not.

Example 3: Given the following functions, write the equation of the inverse. Determine if the function is one-to-one. If it is not, restrict the domain to find an interval on which the function is one-to-one.

1. $f(x) = -4x + 8$	2. $f(x) = (x-2)^2 + 3$	3. $f(x) = 2\sqrt{x-1}$	4. $f(x) = (x-2)^3 + 4$
$x = -4y+8$	$x = (y-2)^2+3$	$x = 2\sqrt{y-1}$	$x = (y-2)^3+4$
$\frac{x-8}{-4} = y$	$\pm\sqrt{x-3}+2 = y$	$\left(\frac{x}{2}\right)^2+1 = y$	$\sqrt[3]{x-4}+2 = y$
yes, one-to-one	Restrict domain either $(-\infty, 2)$ OR $(2, \infty)$	Restrict domain $(-\infty, 0)$	yes, one-to-one

Explain your method on how you found the inverse equations.

switch x and y, solve for y.

What connections do you see between finding the inverse numerically (example 1), graphically (example 2), and algebraically (example 3)?

your own answer {each time you are switching x and y}

Verifying Inverses

There is a natural connection between inverses and function composition that gives further insight into what an inverse actually does: It "undoes" the action of the original function. This idea leads to the following rule:

A function f is one-to-one with inverse function g if an only if:

$f(g(x) = x$, for every x in the domain of g, and

$g(f(x)) = x$, for every x in the domain of f

Use the next example that is completed for you to help finish up your packet!

Show algebraically that $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$ are inverse functions.

$$f(g(x) = f(\sqrt[3]{x-1}) = (\sqrt[3]{x-1})^3 + 1 = x - 1 + 1 = x$$

$$g(f(x)) = g(x^3+1) = \sqrt[3]{x^3+1-1} = \sqrt[3]{x^3} = x$$

Since these equations are true for all x, the Inverse Composition Rule guarantees that f and g are inverses.

Confirm that f and g are inverses by showing that $f(g(x)) = x$ and $g(f(x)) = x$

$f(x) = 3x - 2$ and $g(x) = \frac{x+2}{3}$

$$f(g(x)) = 3\left(\frac{x+2}{3}\right) - 2$$
$$= \frac{\cancel{3}(x+2)}{\cancel{3}} - 2$$
$$= x + \cancel{2} - \cancel{2}$$
$$= x \checkmark$$

$\boxed{\text{yes}}$

$$g(f(x)) = \frac{(3x-2)+2}{3}$$
$$= \frac{3x - \cancel{2} + \cancel{2}}{3} = \frac{\cancel{3}x}{\cancel{3}} = x \checkmark$$

$f(x) = \frac{x+1}{x}$ and $g(x) = \frac{1}{x-1}$

$$f(g(x)) = \frac{\left(\frac{1}{x-1}\right)+1}{\frac{1}{x-1}} = \frac{\frac{\cancel{1}+x-\cancel{1}}{x-1}}{\frac{1}{x-1}} = \frac{\frac{x}{\cancel{x-1}}}{\frac{1}{\cancel{x-1}}} = \frac{x}{1} = x \checkmark$$

$$g(f(x)) = \frac{1}{\frac{x+1}{x} - 1} = \frac{1}{\frac{\cancel{x}+1-\cancel{x}}{x}} = \frac{1}{\frac{1}{x}} \cdot \frac{x}{1} = x \checkmark$$

$\boxed{\text{yes}}$

$f(x) = \frac{x+3}{x-2}$ and $g(x) = \frac{2x+3}{x-1}$

$$f(g(x)) = \frac{\frac{2x+3}{x-1} + 3}{\frac{2x+3}{x-1} - 2} = \frac{\frac{2x+\cancel{3}+3x-\cancel{3}}{\cancel{x-1}}}{\frac{\cancel{2x}+3-\cancel{2x}+2}{\cancel{x-1}}} = \frac{\cancel{5}x}{\cancel{5}} = x \checkmark$$

$\boxed{\text{yes}}$

$$g(f(x)) = \frac{2\left(\frac{x+3}{x-2}\right)+3}{\frac{x+3}{x-2} - 1} = \frac{\frac{2x+\cancel{6}+3x-\cancel{6}}{\cancel{x-2}}}{\frac{\cancel{x}+3-\cancel{x}+2}{\cancel{x-2}}} = \frac{\cancel{5}x}{\cancel{5}} = x \checkmark$$